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1983 J. Phys. A: Math. Gen. 16 L1

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LETTER TO THE EDITOR

**Generalised coherent states and group representations on Hilbert spaces of analytic functions**

Christopher C Gerry

Division of Science and Mathematics, University of Minnesota, Morris, Minnesota 56267, USA

Received 20 October 1982

**Abstract.** Using the example for the group  $SU(1,1)$ , we obtain the basis functions of a Hilbert space of analytic functions from the Perelomov definition of generalised coherent states. The Lie algebra in this space has the form of a Holstein-Primakoff representation appropriate for  $SU(1, 1)$ .

As pointed out by Bargmann (1970), representations of Lie groups can be constructed by realising the elements of the Lie algebra as bilinear products of the boson annihilation and creation operators  $a$  and  $a^\dagger$ . With these boson operators represented by operators defined over a Hilbert space of entire analytic functions, the Bargmann-Hilbert space  $\mathcal{H}_B$ , one can also construct group representations as Hilbert spaces of entire analytic functions. As is well known, the aforementioned Hilbert space  $\mathcal{H}_B$  is closely related to the coherent states associated with the canonical algebra of  $a$  and  $a^\dagger$  operators and the eigenstates  $|n\rangle$  of the number operator  $N = a^\dagger a$  (Schweber 1962).

On the other hand, generalised coherent states for an arbitrary Lie group have been considered by Perelomov (1972). One might expect that an alternative way of constructing a representation of a Lie group on a Hilbert space of analytic functions is to exploit the generalised coherent states in a fashion similar to the ordinary coherent states. In this letter we shall indicate how this is possible for the group  $SU(1, 1) \sim SO(2, 1) \sim Sp(2)$ . We show that in this Hilbert space of analytic functions, which we shall call Perelomov-Hilbert space  $\mathcal{H}_P$ , the  $SU(1, 1)$  group generators have the form of a Holstein-Primakoff (1940) representation.

The  $SU(1, 1)$  Lie algebra consists of the three elements  $\{K_0, K_+, K_-\}$  which satisfy the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \tag{1}$$

The Casimir invariant is

$$Q = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) \tag{2}$$

whose eigenvalues we shall denote as  $k(k-1)$ . We consider here only the representation known as the positive discrete series  $\mathcal{D}^+(k)$  for which the operator  $K_0$  is diagonal,

$$K_0|k, n\rangle = (k+n)|k, n\rangle, \tag{3}$$

where  $n = 0, 1, 2, \dots$  and  $k > 0$ . The basis  $\{|k, n\rangle\}$  is complete and satisfies the orthonormality condition  $\langle k, m|k, n\rangle = \delta_{mn}$ . The state  $|k, n\rangle$  is generated from the

'ground state'  $|k, 0\rangle$  through the relation

$$|k, n\rangle = [\Gamma(2k)/n! \Gamma(n+2k)]^{1/2} (K_+)^n |k, 0\rangle \quad (4)$$

(Barut and Fronsda1 1965).

According to Perelomov (1972), coherent states for the  $\mathcal{D}^+(k)$  representation are given by

$$|\xi, k\rangle = D(\alpha)|k, 0\rangle \quad (5)$$

where

$$D(\alpha) = \exp(\alpha K_+ - \alpha^* K_-) \quad (6)$$

$$= \exp(\xi K_+) \exp(\beta K_0) \exp(\gamma K_-) \quad (7)$$

where  $\alpha = -(\tau/2) e^{-i\varphi}$ ,  $\xi = -\tanh(\tau/2) e^{-i\varphi}$ ,  $\beta = \ln(1 - |\xi|^2)$  and  $\gamma = -\xi^*$ . Expanding the exponential in (7) and using the expression of equation (4), we have from (5)

$$|\xi, k\rangle = (1 - |\xi|^2)^k \sum_{n=0}^{\infty} \left( \frac{\Gamma(n+2k)}{n! \Gamma(2k)} \right)^{1/2} \xi^n |k, n\rangle. \quad (8)$$

The reproducing kernel for these states is

$$K(\xi', \xi; k) = \langle \xi', k | \xi, k \rangle = (1 - |\xi'|^2)^k (1 - |\xi|^2)^k (1 - \xi'^* \xi)^{-2k} \quad (9)$$

and unity is resolved as

$$I = \frac{2k-1}{\pi} \int \frac{d^2 \xi}{(1 - |\xi|^2)^2} |\xi, k\rangle \langle \xi, k|. \quad (10)$$

Now from equation (10) and the orthonormality of the states  $\{|k, n\rangle\}$  we have

$$\delta_{mn} = \langle k, m | I | k, n \rangle = \frac{2k-1}{\pi} \int \frac{d^2 \xi}{(1 - |\xi|^2)^2} \langle k, m | \xi, k \rangle \langle \xi, k | k, n \rangle. \quad (11)$$

From equation (8) we have

$$\langle k, n | \xi, k \rangle = (1 - |\xi|^2)^k [\Gamma(n+2k)/n! \Gamma(2k)]^{1/2} \xi^n. \quad (12)$$

We define the functions  $u_{nk}(\xi) = A_{nk} \xi^n$ , where

$$A_{nk} = [\Gamma(n+2k)/n! \Gamma(2k)]^{1/2}$$

such that from equations (11) and (12) we have

$$\delta_{mn} = \int d\mu_k(\xi) u_{nk}^*(\xi) u_{mk}(\xi) \quad (13)$$

where

$$d\mu_k(\xi) = [(2k-1)/\pi] d^2 \xi (1 - |\xi|^2)^{2k-2}. \quad (14)$$

We shall take the functions  $u_{nk}(\xi)$  as an orthonormal basis in the space  $\mathcal{H}_P$  with the measure of equation (14).

These functions  $u_{nk}(\xi)$  provide a representation of  $SU(1, 1)$  such that  $\mathcal{H}_P \sim \mathcal{D}^+(k)$ . To see this we write the operator  $K_0$ , for a particular  $k$ , as

$$K_0^{(k)} = \xi d/d\xi + k \quad (15)$$

such that

$$K_0^{(k)} u_{nk}(\xi) = (n+k) u_{nk}(\xi). \quad (16)$$

Since  $[d/d\xi, \xi] = 1$ , we have  $d/d\xi \sim b$ ,  $\xi \sim b^+$  where  $b$  and  $b^+$  are boson operators defined in an auxiliary Fock space,  $\mathcal{F}_n$ . Thus we may write equation (15) in this space as

$$K_0^{(k)} = b^+ b + k. \quad (17)$$

The  $SU(1, 1)$  Lie algebra may be completed in  $\mathcal{F}_n$  by writing

$$K_+^{(k)} = b^+ (2k + b^+ b)^{1/2}, \quad (18)$$

$$K_-^{(k)} = (2k + b^+ b)^{1/2} b. \quad (19)$$

Equations (17)–(19) constitute the Holstein–Primakoff representation of  $SU(1, 1)$ . Based on the fact that  $SU(1, 1)$  for  $k = \frac{1}{2}(l + N/2)$  is a dynamical group for the  $N$ -dimensional isotropic harmonic oscillator, Mlodinow and Papanicolaou (1980) have used this Holstein–Primakoff representation to develop a large  $N$  expansion in quantum mechanics for a class of potentials of the form  $V = \alpha r^2 + \beta r^{2\nu}$ . Within a given  $l$ -sector, fairly accurate results are obtained for the ground states. (On the other hand, from the path integral in the  $SU(1, 1)$  coherent state representation (Gerry 1982, Gerry and Silverman 1982) we have recently obtained the large  $N$  limit as a semi-classical limit (Gerry *et al* 1982). From this we obtain a phase integral quantisation rule which gives all the energy levels.)

More details will be presented elsewhere.

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